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Unoriented D-brane instantons vs heterotic worldsheet instantons

Massimo Bianchi[abc] **and Jose F. Morales**[bc]

[a] *CERN TH Division, 1211 Geneva, Switzerland*
[b] *Dipartimento di Fisica, Universitá di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy*
[c] *I.N.F.N. Sezione di Roma II, via della Ricerca Scientifica, 00133 Roma, Italy*
E-mail: bianchi@roma2.infn.it, morales@roma2.infn.it

Abstract: We discuss Fermi interactions of four hyperini generated by "stringy" instantons in a Type I / Heterotic dual pair on $T^4/\mathbb{Z}_2$.

Keywords: Intersecting branes models, String Duality, D-branes, Superstrings and Heterotic Strings.

Contents

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1. Introduction

In the last year or so, significant progress has been achieved in the understanding of non-perturbative superpotentials generated by unoriented D-brane instantons in Type I theories and alike [1 − 22]. D-brane instantons are euclidean D-branes entirely wrapped along the internal compactification manifold and intersecting the D-brane gauge theory at a point in spacetime. The instanton dynamics can be efficiently described by effective gauge theories governing the low energy interactions of open strings ending on the Dp-brane instanton [23, 24]. The case when the brane instanton is parallel to the brane hosting the gauge theory along the internal space can be associated to YM instantons with self-dual field strength. We refer to them as "gauge instantons". When the instanton and gauge theory branes wrap different cycles in the internal manifold they are called "exotic" or "stringy" since a clear field theory interpretation is still missing.

Only under very special conditions a D-brane instanton can generate a superpotential in four dimensions. First, orbifold and orientifold actions should be combined in such a way that out of the original instanton fermionic zero modes only two, to be identified as the superspace fermionic coordinates θ_α, survive the projections. Second, the superpotential should transform in such a way as to balance the non-trivial transformation properties of the classical instanton action under all U(1)'s coming from D-branes intersecting the instanton. The resulting superpotential violates the U(1) symmetries under which the instanton is charged, leading to couplings in the low energy action that are otherwise forbidden at the perturbative level. In [1 − 22], brane-worlds exposing both gauge and exotic instanton generated superpotentials have been worked out.

D-brane instantons are not new to the string literature. They showed up in the study of string dualities where brane and worldsheet instantons were shown to be related to each other via strong/weak coupling duality[1]. The archetypical example is the computation of thresholds corrections to BPS saturated couplings, such as F^4 terms, in toroidal compactifications of Type I / Heterotic string theories. The F^4 coupling is corrected by ED1 branes in Type I theory and worldsheet instantons in the Heterotic string and the instanton sums have been shown to precisely match on the two sides [28 − 30]. The instanton dynamics was described in terms of the effective gauge theory governing the interactions of ED1-D9 strings [31 − 34]. Unlike in the previously discussed exotic situation, in this case brane instantons do not violate perturbative symmetries of the theory and the F^4-couplings are corrected both at the perturbative and non-perturbative level. Similar results have been found for $\mathcal{R}^2$-terms in type I theory on T^6, where the infinite tower of ED5-instanton corrections can be rephrased in terms of worldsheet instantons in type IIA theory on $K3 \times T^2$ [35].

It is natural to ask whether superpotentials generated at the non-perturbative level by stringy instantons have a worldsheet instanton counterpart. Aim of this work is to consider the simplest setting where such a map can be built. We will exploit Type I /Heterotic duality to rephrase exotic ED1-instanton corrections in terms of worldsheet instantons in the Heterotic string. We believe that a clear dictionary between the two will set the rules for exotic multi-instanton calculus on firmer grounds.

We start by considering brane / worldsheet instanton corrections in a $\mathcal{N} = (1, 0)$ Heterotic / Type I dual pair in $D = 6$ after compactification on $T^4/\mathbb{Z}_2$. The present analysis has been partly inspired by the elegant test of the duality between Heterotic string on T^4 and Type IIA on $K3$ performed by Kiritsis, Obers and Pioline [30]. The Type I vacuum is the $\mathrm{U}(16) \times \mathrm{U}(16)$ model constructed in [36] and rederived in [37], see also [38 − 44]. In the heterotic dual, the SO(32) group is broken by the orbifold projection to U(16). To achieve exact duality, the Type I D5-branes, giving rise to the second factor in the Chan-Paton group, are to be democratically distributed over the sixteen fixed points, breaking the D5-brane gauge group U(16) down to $\mathrm{U}(1)^{16}$. The 16 U(1) photons then become massive by eating twisted matter leading to a massless spectrum that matches precisely the one of the Heterotic string [44]. We will consider quartic couplings involving hypermultiplets coming from the twisted sectors in the heterotic string. When the four hypers are located at different fixed points this coupling is absent in perturbation theory but is generated by ED1 / worldsheet instantons wrapping two-cycles intersecting the four fixed points.

[1]For review see e.g. [25 − 27]

Sector	U(16) representations
Untwisted	$\mathbf{1_0}$ G $+\mathbf{1_0}$ T$+\mathbf{256_0}$ V$+(4\times\mathbf{1_0}+\mathbf{120_2}+\mathbf{120^*_{-2}})$ H
Twisted	$16\times\mathbf{16_{-3}}$ H

Table 1: Massless spectrum of heterotic string on $T^4/\mathbb{Z}_2$. G,T, V,H refers to gravity, tensor, vector and hyper multiplets of $\mathcal{N}=(1,0)$ $D=6$ supersymmetry.

It would be nice to extend our analysis to Heterotic / Type I dual pairs with $\mathcal{N}=1$ supersymmetry in $D=4$ dimensions. In particular the $T^6/\mathbb{Z}_3$ orbifold of [45], reconsidered from the D-brane instanton perspective in [4, 7], would be the first natural candidate to focus on, possibly after inclusion of Wilson lines [46, 47]. The results for Type I / Heterotic dual pairs on freely acting orbifolds [16] go along this line.

2. Type I/heterotic duals on $T^4/\mathbb{Z}_2$

The first instance of a rather non-trivial yet workable Heterotic / Type I dual pair is the U(16) model with $\mathcal{N}=(1,0)$ supersymmetry in $D=6$, emerging from a $T^4/\mathbb{Z}_2$ orbifold compactification. On the Type I side the model was originally discussed among many others in [36] and was then rediscovered in terms of D-branes and Ω-planes in [37]. The combined effect of orbifold and unoriented projections requires the presence of 32 D5-branes in addition to the usual 32 D9-branes. At the maximally symmetric point, where all the D5-branes are on top of an orientifold O5-plane and no Wilson lines are turned on the D9's, the gauge group is $\mathrm{U}(16)\times\mathrm{U}(16)$. With respect to many models of the kind [36, 38 – 43] this Type I model is peculiar in that it has only one tensor multiplet, whose scalar component determines the volume of $T^4/\mathbb{Z}_2$, that can be related via Type I / Heterotic duality to the unique tensor multiplet, containing the dilaton in the heterotic side [2].

Besides its geometrical action, the $\mathbb{Z}_2$ orbifold acts on the 32 heterotic fermions $(\lambda^u,\lambda^{\bar{u}})$, $u,\bar{u}=1,\dots 16$ (accounting for the world-sheet current algebra) as $(i^{16},(-i)^{16})$. This action effectively breaks SO(32) to U(16). The resulting spectrum is summarized in table 1. We recall that a hyper H in a representation $\mathbf{R}$ consists of two half-hypers (each containing a complex boson and a chiral fermion) transforming in the representation $\mathbf{R}+\mathbf{R}^*$. Notice that the absence of twisted singlets, the would-be blowing up modes, implies that one cannot resolve the (hyperkahler) orbifold singularity and reach a smooth K3 compactification without at the same time breaking U(16) at least to SU(15) by giving VEV to the charged twisted hypers [48].

At the maximally symmetric point, the Type I spectrum looks rather different (see table 2). In order to make contact with the heterotic description, one has to start with the 16 D5-branes democratically distributed among the 16 fixed points thus breaking U(16) down to $\mathrm{U}(1)^{16}$. The necessary 240 = (256 -16) hypers are provided by the open string hypers in the $(\mathbf{120_2}+\mathbf{120^*_{-2}},\mathbf{1_0})$. It is important to notice that the two vacua belong

[2] Additional tensor multiplets can be contributed by heterotic M5-branes, i.e. pointlike $E(8)$ instantons, which do not admit a fullfledged worldsheet description. Pointlike SO(32) instantons support vector multiplets very much like pointlike D5-branes in Type I, i.e. inside D9-branes.

Sector	$\mathrm{U}(16)\times\mathrm{U}(16)$ representations
Untwisted closed	$(\mathbf{1_0},\mathbf{1_0})$ G $+(\mathbf{1_0},\mathbf{1_0})$ T$+4\,(\mathbf{1_0},\mathbf{1_0})$ H
Twisted closed	$16\,(\mathbf{1_0},\mathbf{1_0})$ H
99-open strings	$(\mathbf{1_0},\mathbf{256_0})$ V$+(\mathbf{1_0},\mathbf{120_2}+\mathbf{120^*_{-2}})$ H
55-open strings	$(\mathbf{256_0},\mathbf{1_0})$ V$+(\mathbf{120_2}+\mathbf{120^*_{-2}},\mathbf{1_0})$ H
59-open strings	$(\mathbf{16^*_{-1}},\mathbf{16_1})$ H

Table 2: Massless spectrum of Type I on $T^4/\mathbb{Z}_2$ at the $\mathrm{U}(16)\times\mathrm{U}(16)$ point. G,T,V,H refers to gravity, tensor, vector and hyper multiplets of $\mathcal{N}=1$ $D=6$ supersymmetry. Underlined fields are Higgsed away when D5-branes are democratically distributed among the 16 fixed points .

to disconnected components of the type I moduli space since there is no continuous deformation connecting the $\mathrm{U}(16)$ and $\mathrm{U}(1)^{16}$ D5-brane distributions[3]. The 16 $\mathrm{U}(1)$'s are anomalous. The corresponding vector multiplets get massive by eating as many 'neutral' closed string hypers in a supersymmetric fashion. The surviving massless spectrum precisely matches the heterotic spectrum in table 1 once the surviving (anomalous) $\mathrm{U}(1)$ is identified with $Q_H = Q_9 + 4\sum_i Q_5^i$ in such a way that the charge of the **16** hypers is $1+4(-1)=-3$. The reducible $\mathrm{U}(1)$ anomaly is disposed of by means of the $D=6$ version of the GS mechanism [49−51].

We will consider instanton generated chiral couplings in the $\mathcal{N}=1$ low energy effective action. The main working example is the four-hyperini Fermi interaction[4]

$$
\begin{aligned}
S_{4Fermi} &= \int d^6x d^4\theta\, W_{f_1f_2f_3f_4}(q) \prod_{i=1}^{4} H_{16,f_i} \qquad (2.1)\\
&= \int d^6x\, W_{f_1f_2f_3f_4}(q)\, \epsilon^{a_1\ldots a_4}\, \zeta^{u_1}_{a_1,f_1}\, \zeta^{\bar u_2}_{a_2,f_2}\, \zeta^{u_3}_{a_3,f_3}\, \zeta^{\bar u_4}_{a_4,f_4} \delta_{u_1\bar u_2}\delta_{u_3\bar u_4} + \ldots
\end{aligned}
$$

with H_{16,f_i} the hypermultiplet superfield in the $\mathbf{16}_{-3}+\mathbf{16}^*{}_{+3}$ of $\mathrm{U}(16)$ localized at the fixed point f_i and $W_{f_1f_2f_3f_4}(q)$ encoding the dependence on the neutral moduli q, parametrizing Kähler and complex structure deformations of $T^4/\mathbb{Z}_2$. Indices $u,\bar u = 1,\ldots 16$, $a=1,\ldots 4$ run over the fundamental and spinor representation of $\mathrm{U}(16)$ and $\mathrm{SO}(5,1)$ respectively. Dots in (2.2) refer to other four-fermi interactions obtained by permutations of the $u_i,\bar u_i$ superscripts of $\zeta^{u_i}_{a_i,f_i}$ and supersymmetry related terms. The four-fermi coupling in (2.2) can be thought of as the supersymmetric partner of a two-derivative four-scalar term describing the hypermultiplet metric [52].

If the four fixed points are chosen to be different from one another, the coupling (2.2) is absent to any order in perturbation theory since twisted fields lying at different fixed points do not interact perturbatively. Such a term can instead be generated via ED1-brane or worldsheet instantons connecting the four fixed points. The contributions will be exponentially suppressed with the area of the $S^2 \sim T^2/\mathbb{Z}_2$ cycle wrapped by the instanton.

[3] We thank I. Antoniadis for drawing our attention onto this subtle point and A. Uranga for clarifying it.

[4] The Fierz identity $(\gamma_\mu)^{ab}(\gamma^\mu)^{cd} = \epsilon^{abcd}$ in $D=6$ explains the nomenclature.

In the next sections we will determine the moduli dependence $W_{f_1 f_2 f_3 f_4}(q)$ of the four hyperini coupling from string amplitudes in the Heterotic and Type I theories.

Before entering the details of the computation, let us comment on which kind of corrections one should expect in the two descriptions. Moduli coming from the untwisted sector in Type I / Heterotic theory on $T^4/\mathbb{Z}_2$ span the symmetric space $[\mathrm{SO}(4,4)/\,\mathrm{SO}(4)\times\mathrm{SO}(4)]\times\mathrm{SO}(1,1)$. The $\mathrm{SO}(1,1)$ factor accounts for the scalar in the tensor multiplet. In $D=6$, Heterotic / Type I duality amounts to the following field identifications [45]

$$\phi_{_H} = \omega_{_I} \qquad , \qquad \phi_{_I} = \omega_{_H} \tag{2.2}$$

where ϕ and ω denote the dilaton and volume modulus respectively. Supersymmetry implies that there is no neutral couplings between vectors and hypers [53, 54]. The gauge coupling can only depend (linearly) on the scalar in the unique tensor multiplet $\phi_{_H}=\omega_{_I}$, while $\phi_{_I}=\omega_{_H}$ belongs to a neutral hyper. For this reasons the hypermultiplet geometry should be tree-level exact in the heterotic description, but may receive worldsheet instanton corrections. On the other hand, in the Type I description it can receive both perturbative and non-perturbative corrections.

3. Heterotic amplitude

In this section we compute the four-hyperini Fermi interaction in the heterotic description. The relevant heterotic string amplitude is

$$\begin{aligned}\mathcal{A}^{Het}_{4Fermi}(s,t) &= \int d^2 z_3 \langle c\tilde{c}V^{\zeta}_{\mathbf{16}}(z_1,\bar{z}_1)c\tilde{c}V^{\zeta}_{\mathbf{16}^*}(z_2,\bar{z}_2)V^{\zeta}_{\mathbf{16}}(z_3,\bar{z}_3)c\tilde{c}V^{\zeta}_{\mathbf{16}^*}(z_4,\bar{z}_4)\rangle \\ &= \zeta^{u_1}_{a_1,f_1}\zeta^{\bar{u}_2}_{a_2,f_2}\zeta^{u_3}_{a_3,f_3}\zeta^{\bar{u}_4}_{a_4,f_4}\epsilon^{a_1\ldots a_4}\left[\delta_{u_1\bar{u}_2}\delta_{u_3\bar{u}_4}\mathcal{A}^{(12|34)}_{f_i} + \delta_{u_1\bar{u}_4}\delta_{u_3\bar{u}_2}\mathcal{A}^{(14|32)}_{f_i}\right]\end{aligned} \tag{3.1}$$

Notice that $\mathcal{A}^{(12|34)}_{f_i}$ and $\mathcal{A}^{(14|32)}_{f_i}$ are related by a simple relabelling of the fixed points f_i's therefore we can restrict our attention onto the amplitude $\mathcal{A}^{(12|34)}_{f_i}$ with color structure $\delta_{u_1\bar{u}_2}\delta_{u_3\bar{u}_4}$.

The heterotic string vertex operators read

$$\begin{aligned}V^{\zeta}_{\mathbf{16}} &= \zeta^{u}_{a}(p)\, e^{-\varphi/2}\,\sigma_f\, S^a\tilde{\Sigma}_u\, e^{ipX} \\ V^{\zeta}_{\mathbf{16}^*} &= \zeta^{\bar{u}}_{a}(p)\, e^{-\varphi/2}\,\sigma_f\, S^a\tilde{\Sigma}_{\bar{u}}\, e^{ipX}\end{aligned} \tag{3.2}$$

with

$$\tilde{\Sigma}_u = \prod_{v=1}^{16} e^{i(-\frac{1}{4}+\delta_{uv})\,\varphi_v} \qquad \tilde{\Sigma}_{\bar{u}} = \prod_{v=1}^{16} e^{i(\frac{1}{4}-\delta_{uv})\,\varphi_v} \tag{3.3}$$

The various fields entering the string vertices are defined as follows. S^a are $\mathrm{SO}(5,1)$ spin fields, φ and φ_u the bosonization of the superghost and $\mathrm{SO}(32)$ gauge fermions respectively and σ_f is the bosonic $\mathbb{Z}_2$-twist field. The contribution to the conformal dimension $(h,\bar{h})$ of the string vertex operators from S^a, $e^{-\varphi/2}$, $\tilde{\Sigma}_{u,\bar{u}}$ and σ_f sum up to (1,1), *viz.*

$$(h,\bar{h}) = \left(\frac{3}{8},0\right) + \left(\frac{3}{8},0\right) + \left(0,\frac{3}{4}\right) + \left(\frac{1}{4},\frac{1}{4}\right) = (1,1) \quad , \tag{3.4}$$

as expected for a massless field. Finally $c, \tilde{c}$ are the ghost associated to the $\mathrm{SL}(2,\mathbb{C})$ invariance that allows to fix the positions z_1, z_2, z_4 of three vertices. The string amplitude will then depend on the $\mathrm{SL}(2,\mathbb{C})$ invariant cross ratio z defined as

$$z = \frac{z_{12}z_{34}}{z_{13}z_{24}} \tag{3.5}$$

with $z_{ij} = z_i - z_j$. The worldsheet correlators needed to compute the heterotic string amplitude read

$$\begin{aligned}
\frac{\prod_i d^2 z_i}{dV_{\mathrm{SL}(2,\mathbb{C})}} &= d^2 z_3 \langle c\tilde{c}(z_1) c\tilde{c}(z_2) c\tilde{c}(z_4)\rangle = d^2 z_3 |z_{12}z_{14}z_{24}|^2 = d^2 z |z_{13}z_{24}|^4 \\
\langle \prod_{i=1}^4 e^{ik_i X}(z_i)\rangle &= \prod_{i<j} z_{ij}^{2\alpha' k_i k_j} = |z|^{\alpha' s}\, |1-z|^{\alpha' t} \\
\langle \prod_{i=1}^4 S^{a_i} e^{-\varphi/2}(z_i)\rangle &= \epsilon^{a_1\ldots a_4} \prod_{i<j} z_{ij}^{-\frac{1}{2}} \\
\langle\, \tilde{\Sigma}_{u_1}(\bar{z}_1)\, \tilde{\Sigma}_{\bar{u}_2}(\bar{z}_2)\, \tilde{\Sigma}_{\bar{u}_3}(\bar{z}_3)\, \tilde{\Sigma}_{\bar{u}_4}(\bar{z}_4)\rangle &= \prod_{i<j} \bar{z}_{ij}^{-\frac{1}{2}} \left(\frac{1}{\bar{z}}\, \delta_{u_1\bar{u}_2}\delta_{u_3\bar{u}_4} + \frac{1}{1-\bar{z}}\, \delta_{u_1\bar{u}_4}\delta_{u_3\bar{u}_2} \right)
\end{aligned} \tag{3.6}$$

Finally the four $\mathbb{Z}_2$-twist field correlator is given by [55]:

$$\mathcal{C}_{4\sigma} = \langle \prod_{i=1}^4 \sigma_{f_i}(z_i, \bar{z}_i)\rangle = \mathcal{C}_{qu}\, \Lambda_{cl} \begin{bmatrix} \vec{f}_{12} \\ \vec{f}_{13} \end{bmatrix} \tag{3.7}$$

with $\mathcal{C}_{qu}$ and Λ_{cl} the quantum and classical parts of the correlator respectively. The quantum part $\mathcal{C}_{qu}$ is independent of the location of the twist fields while the classical part $\Lambda_{cl} = \sum e^{-S_{\rm inst}}$ sums over all possible wrapping of the string world-sheet passing through the 4 fixed points.

We denote by $\vec{f}_i$ the locations of the four fixed points

$$\vec{f}_i = \frac{1}{2}(\epsilon_i^1, \epsilon_i^2, \epsilon_i^3, \epsilon_i^4) \qquad \epsilon_i^a = 0, 1 \tag{3.8}$$

and by $\vec{f}_{ij} = \vec{f}_i - \vec{f}_j$ their relative positions. In order to get a non-trivial coupling the $\vec{f}_i$ should satisfy the selection rule

$$\sum_{i=1}^4 \vec{f}_i = \vec{0} \mod 2 \tag{3.9}$$

The two pieces in (3.7) can be written in terms of the Teichmüller parameter $\tau(z)$ of the torus doubly covering the sphere with two $\mathbb{Z}_2$ branch cuts. The relation between the cross ratio z and $\tau(z)$ is coded in

$$z = \frac{\vartheta_3^4(\tau)}{\vartheta_4^4(\tau)} \tag{3.10}$$

The quantum and classical parts of the correlator read [55]

$$\mathcal{C}_{qu} = 2^{-\frac{8}{3}} \prod_{i<j} |z_{ij}|^{-\frac{1}{3}} \tau_2^{-2} |\eta(\tau)|^{-8} \tag{3.11}$$

$$\Lambda_{cl} \begin{bmatrix} \vec{f}_{12} \\ \vec{f}_{13} \end{bmatrix} = \sqrt{\det G} \sum_{\substack{\vec{n} \in \mathbb{Z}+\vec{f} \\ \vec{m} \in \mathbb{Z}+\vec{h}}} e^{-\frac{\pi}{\tau_2}(\vec{n}\tau+\vec{m})^t G(\vec{n}\bar{\tau}+\vec{m})+2\pi i \vec{n} B \vec{m}}$$

with G and B denoting the metric and antisymmetric tensor on T^4. Plugging (3.6) and (3.11) into (3.1) and using

$$|z_{13} z_{24}|^4 \prod_{i<j} |z_{ij}|^{-\frac{4}{3}} = |z(1-z)|^{-\frac{4}{3}} \tag{3.12}$$

one finds

$$\mathcal{A}_{f_i}^{(12|34)}(s,t) = \int \frac{d^2 z}{\tau_2^2 |\eta(\tau)|^8} |z|^{\alpha' s - \frac{4}{3}} |1-z|^{\alpha' t - \frac{4}{3}} \frac{1}{\bar{z}} \Lambda_{cl} \begin{bmatrix} \vec{f}_{12} \\ \vec{f}_{13} \end{bmatrix} \tag{3.13}$$

Switching to the torus measure via

$$\frac{dz}{z} = i\pi \vartheta_2^4(\tau) d\tau \tag{3.14}$$

and sending $s, t \to 0$[5] one finds

$$\mathcal{A}_{f_i}^{(12|34)}(0,0) = \int_{\mathcal{F}_2} \frac{d^2\tau}{\tau_2^2} \frac{\bar{\vartheta}_4^4}{\bar{\vartheta}_3^4} \Lambda_{cl} \begin{bmatrix} \vec{f}_{12} \\ \vec{f}_{13} \end{bmatrix} \tag{3.15}$$

The integral runs over the fundamental domain $\mathcal{F}_2$ of the index six subgroup Γ_2 of $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, defined as the group of modular transformations leaving invariant $\vartheta_2^4, \vartheta_3^4, \vartheta_4^4$. Interestingly, as can be seen from (3.15), the four hyperini coupling receives contributions only from ground states of the right moving supersymmetric sector of the string. Indeed one can think of (3.15) as a "1/2 BPS saturated" partition function counting $\mathbb{Z}_2$-invariant wrappings of the worldsheet instanton along $T^4/\mathbb{Z}_2$ passing through the four fixed points $\vec{f}_i$ and carrying the right U(16) charges to couple to the **16** hyperini. The fact that only BPS worldsheet instantons contribute to the four-hyperini amplitude is not surprising since only states preserving half of the eight supercharges of the theory can correct a chiral term in the $D = 6$ $\mathcal{N} = (1,0)$ low energy effective action.

3.1 The integral

Integrals of modular invariant functions times unshifted lattice sums over the fundamental domain of the modular group $\mathrm{SL}(2,\mathbb{Z})$ where first computed by Dixon, Kaplunovsky and Louis [56]. Their classic results were later on generalized to integrals of Γ_2-modular forms and shifted lattice sums [57 – 61].

In this section we closely follow the strategy advocated in [30] to evaluate the integral

$$\mathcal{I} \begin{bmatrix} \vec{f} \\ \vec{h} \end{bmatrix} = \int_{\mathcal{F}_2} \frac{d^2\tau}{\tau_2^2} \Phi(\bar{\tau}) \Lambda_{cl} \begin{bmatrix} \vec{f} \\ \vec{h} \end{bmatrix} \qquad \Phi(\bar{\tau}) = \frac{\bar{\vartheta}_4^4}{\bar{\vartheta}_3^4} \tag{3.16}$$

[5] Regularization of IR divergences is understood when some fixed points coincide.

G	$\Lambda_{\text{invariant}}$	$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$	generators	$\mathcal{F}_G = \mathbb{C}^+/G$
Γ	$\Lambda[^{\vec{0}}_{\vec{0}}]$	$\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}$	S, T	$\mathcal{F}$
Γ_2^+	$\Lambda[^{\vec{f}}_{\vec{0}}]$	$\begin{pmatrix} 2\mathbb{Z}+1 & 2\mathbb{Z} \\ \mathbb{Z} & 2\mathbb{Z}+1 \end{pmatrix}$	T^2, STS	$\{1, S, T\}\mathcal{F}$
Γ_2^-	$\Lambda[^{\vec{0}}_{\vec{h}}]$	$\begin{pmatrix} 2\mathbb{Z}+1 & \mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z}+1 \end{pmatrix}$	T, ST^2S	$\{1, S, ST\}\mathcal{F}$
Γ_2^0	$\Lambda[^{\vec{f}}_{\vec{f}}]$	$\begin{pmatrix} 2\mathbb{Z}+1 & 2\mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z}+1 \end{pmatrix}, \begin{pmatrix} 2\mathbb{Z} & 2\mathbb{Z}+1 \\ 2\mathbb{Z}+1 & 2\mathbb{Z} \end{pmatrix}$	S, T^2	$\{1, T, TS\}\mathcal{F}$
Γ_2	$\Lambda[^{\vec{f}}_{\vec{h}}]$	$\begin{pmatrix} 2\mathbb{Z}+1 & 2\mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z}+1 \end{pmatrix}$	T^2, ST^2S	$\{1, T, S, TS, ST, TST\}\mathcal{F}$

Table 3: Modular group $\Gamma = SL(2,\mathbb{Z})$ and its finite index subgroups of order 2 together with the corresponding invariant lattice sums, matrix representations, generators and fundamental domains.

We start by recalling some basic facts about the modular group $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ and its order 2 finite index subgroups. A more detailed discussion can be found in the appendix.

Γ acts on the upper half-plane $\mathbb{C}^+ = \{\tau_1 \in (-\infty,\infty), \tau_2 \in (0,\infty)\}$ by projective transformations of the modular parameter τ

$$\tau \to \frac{a\tau+b}{c\tau+d} \quad \text{with} \qquad \left\{a,b,c,d \in \mathbb{Z}; \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1\right\} \tag{3.17}$$

The group Γ is generated by the transformations

$$S: \tau \to -1/\tau \quad , \quad T: \tau \to \tau+1 \tag{3.18}$$

under which the shifted latticed sums transform according to

$$\begin{aligned} \Lambda_{cl}\left[^{\vec{f}}_{\vec{h}}\right]\left(-\frac{1}{\tau}\right) &= \Lambda_{cl}\left[^{\vec{h}}_{\vec{f}}\right](\tau) \\ \Lambda_{cl}\left[^{\vec{f}}_{\vec{h}}\right](\tau+1) &= \Lambda_{cl}\left[^{\vec{f}}_{\vec{h}+\vec{f}}\right](\tau) \end{aligned} \tag{3.19}$$

Γ_2 is the subgroup of Γ leaving $\Lambda[^{\vec{f}}_{\vec{h}}]$ invariant for any $f_i, h_i = 0, \frac{1}{2}$. In a similar way $\Gamma_2^+, \Gamma_2^-, \Gamma_2^0$ can be identified as the subgroups of Γ leaving invariant the shifted lattice sums $\Lambda[^{\vec{f}}_{\vec{0}}]$, $\Lambda[^{\vec{0}}_{\vec{h}}]$ and $\Lambda[^{\vec{f}}_{\vec{f}}]$, respectively. The generators of the modular subgroups and their fundamental domains $\mathcal{F}^{0,\pm} = \mathbb{C}^+/\Gamma_2^{0,\pm}$ can be found by restricting the parity of a, b, c, d entering in (3.17). We summarize the results in table 3.

Now we are ready to perform the integral (3.15). Following [30], write the instanton sum as[6]

$$\Lambda\left[^{\vec{f}}_{\vec{h}}\right](\tau,\bar{\tau};G,B) = \sqrt{\det G} \sum_{\substack{\vec{n}\in\mathbb{Z}+\vec{f} \\ \vec{m}\in\mathbb{Z}+\vec{h}}} e^{-\frac{\pi}{\tau_2}(\vec{n}\tau+\vec{m})G(\vec{n}\bar{\tau}+\vec{m})+2\pi i\vec{n}B\vec{m}} \tag{3.20}$$

and introduce the induced metric and antisymmetric tensors

$$\mathcal{G}_{\alpha\beta} = M^i_\alpha G_{ij} M^j_\beta \quad , \quad \mathcal{B}_{\alpha\beta} = M^i_\alpha B_{ij} M^j_\beta \tag{3.21}$$

[6] Here we use the shorthand notation $\vec{m}\, A\, \vec{n} = m_i\, A_{ij}\, n_j$

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where $M^i_\alpha = (n^i, m^i)$. More explicitly

$$\mathcal{G}_{11} = \vec{n}\, G\, \vec{n} \quad , \quad \mathcal{G}_{22} = \vec{m}\, G\, \vec{m} \quad , \quad \mathcal{G}_{12} = \vec{n}\, G\, \vec{m} \quad , \quad \mathcal{B}_{12} = \vec{n}\, B\, \vec{m} \tag{3.22}$$

Also define the induced Kähler $\mathcal{T}(M)$ and complex structures $\mathcal{U}(M)$

$$\begin{aligned} \mathcal{T}(M) &= \left(\mathcal{B}_{12} + i\sqrt{\det \mathcal{G}}\right) \\ \mathcal{U}(M) &= \mathcal{G}_{11}^{-1}\left(\mathcal{G}_{12} + i\sqrt{\det \mathcal{G}}\right) \end{aligned} \tag{3.23}$$

In such a way that the instanton sum becomes

$$\Lambda \begin{bmatrix} \vec{f} \\ \vec{h} \end{bmatrix} = \sqrt{\det G} \sum_{M(\vec{f},\vec{h})} e^{2\pi i \mathcal{T}(M)} e^{-\frac{\pi \mathcal{T}_2(M)}{\tau_2 \mathcal{U}_2(M)}|\tau - \mathcal{U}(M)|^2} \tag{3.24}$$

with the sum running over

$$M = (\vec{n}, \vec{m}) \in (\mathbb{Z}^4 + \vec{f}, \mathbb{Z}^4 + \vec{h}) \tag{3.25}$$

The sum over M can be rewritten as a sum of representatives of Γ_2-orbits and Γ_2-images of the fundamental domain. This exploits the fact that contributions coming from two different M's related by

$$M \to M \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\vec{n}a + \vec{m}c \ , \ \vec{n}b + \vec{m}d\right) \tag{3.26}$$

can be rewritten as integrals of the same representative on two domains related by the corresponding modular transformations on τ. The different orbits are classified by the sub-determinants of M and are indicated as *trivial* orbit for $M = 0$, *degenerate* orbits for $\det M_{ij} = m_i n_j - m_j n_i = 0 \ \forall i, j$ and *non-degenerate* orbits otherwise.

Let us in turn consider the various contributions

3.1.1 Trivial orbit

$M = 0$, only present for $\vec{f} = \vec{h} = 0$, corresponds to the (regulated) contribution of the massless particles exchange. Since $\Lambda \begin{bmatrix} \vec{0} \\ \vec{0} \end{bmatrix}$ is modular invariant one can replace $\Phi(\bar{\tau})$ with its images under Γ/Γ_2. One finds

$$\begin{aligned} \mathcal{I}_{triv} \begin{bmatrix} \vec{0} \\ \vec{0} \end{bmatrix} &= \sqrt{\det G} \int_{\mathcal{F}_2} \frac{d^2\tau}{\tau_2^2} \Phi(\bar{\tau}) = \sqrt{\det G} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \sum_{s=1}^{6} \Phi(\bar{\tau}_s) \\ &= 3\sqrt{\det G} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} = \pi^2 \sqrt{\det G} \end{aligned} \tag{3.27}$$

with τ_s the image of τ under the modular transformations in the quotient Γ/Γ_2 so that

$$\tau_s = (1, S, T, TS, ST, TST)\tau \tag{3.28}$$

The third equality in (3.27) follows from the identity

$$\sum_{s=1}^{6} \Phi(\bar{\tau}_s) = 3 \tag{3.29}$$

that can be easily checked using Jacobi's *aequatio identica satis abstrusa*.

3.1.2 Degenerate orbits.

Degenerate orbits. $\det(M_{ij}) = n_i m_j - n_j m_i = 0\ \forall i,j$ with $M \neq 0$. They are present whenever $f_i h_j - f_j h_i = 0\ \forall i,j$ i.e. only when $\vec{f} = 0$ or $\vec{h} = 0$ or $\vec{f} = \vec{h}$. Since each of these three 'twist structures' is invariant under a larger subgroup $\Gamma_2^{\pm,0}$ (rather than Γ_2) it is convenient to exploit this extra symmetry.

In the case $\vec{f} = 0$, $\Lambda_{cl}[{}^{\vec{0}}_{\vec{h}}]$ is invariant under Γ_2^- and the integral can be written as

$$\mathcal{I}_{\rm deg}\begin{bmatrix}\vec{0}\\ \vec{h}\end{bmatrix} = \int_{\mathcal{F}_2} \frac{d^2\tau}{\tau_2^2}\Phi(\bar{\tau})\Lambda_{\rm deg}\begin{bmatrix}\vec{0}\\ \vec{h}\end{bmatrix} = \int_{\mathcal{F}_2^-} \frac{d^2\tau}{\tau_2^2}[\Phi(\bar{\tau}) + \Phi(\bar{\tau}+1)]\Lambda_{\rm deg}\begin{bmatrix}\vec{0}\\ \vec{h}\end{bmatrix} \tag{3.30}$$

The degeneracy condition implies that elements in this orbit can be written as

$$\vec{n} = c\,\vec{\ell} \qquad \vec{m} = d\,\vec{\ell} \qquad \vec{\ell} \in \mathbb{Z}^4 + \vec{h} \quad c \in 2\mathbb{Z} \quad d \in 2\mathbb{Z}+1 \tag{3.31}$$

Comparing with (3.26) we see that $M = (\vec{n}, \vec{m})$ in (3.31) correspond to Γ_2^--images of the representative

$$M_{\rm deg}^- = (\vec{0}, \vec{\ell}) \tag{3.32}$$

Such representative is invariant under T ($a = b = d = 1$, $c = 0$) and therefore the remaining modular transformations unfold the fundamental domain $\mathcal{F}_2^-$ to the strip $\mathbb{C}^+/T = \{\tau_2 > 0, |\tau_1| < 1/2\}$.

Since with the chosen representative $\Lambda_{\rm deg}\begin{bmatrix}\vec{0}\\ \vec{h}\end{bmatrix}$ is independent of τ_1, the τ_1 integral can be performed immediately and projects onto the constant $d_0 = 2$ in the expansion of $\Phi(\bar{\tau}) + \Phi(\bar{\tau}+1) = \sum_n d_n \bar{q}^n$. Integration over τ_2 then yields

$$\mathcal{I}_{\rm deg}\begin{bmatrix}\vec{0}\\ \vec{h}\end{bmatrix} = 2\sqrt{\det G}\int_0^\infty \frac{d\tau_2}{\tau_2^2}\sum_{\vec{m}\in\mathbb{Z}+\vec{h}} e^{-\frac{\pi}{\tau_2}\vec{m}G\vec{m}} = \frac{2\sqrt{\det G}}{\pi}\sum_{\vec{m}\in\mathbb{Z}+\vec{h}} (\vec{m}G\vec{m})^{-1} \tag{3.33}$$

Although independent of B, this contribution has no counterpart in the type I description at lowest perturbative order (disk). Indeed, as we will see in the next section, there is no room for connected disk amplitudes with such $\delta_{u_1\bar{u}_2}\delta_{f_1f_2}\delta_{u_3\bar{u}_4}\delta_{f_3f_4}$ tensor structure.

In the second case, $\vec{h} = \vec{f}$, $\Lambda_{cl}[{}^{\vec{f}}_{\vec{f}}]$ is invariant under Γ_2^0 that includes S in the quotient Γ_2^0/Γ_2. The relevant integrals are

$$\mathcal{I}_{\rm deg}\begin{bmatrix}\vec{f}\\ \vec{f}\end{bmatrix} = \int_{\mathcal{F}_2} \frac{d^2\tau}{\tau_2^2}\Phi(\bar{\tau})\Lambda_{\rm deg}\begin{bmatrix}\vec{f}\\ \vec{f}\end{bmatrix} = \int_{\mathcal{F}_2^0} \frac{d^2\tau}{\tau_2^2}[\Phi(\bar{\tau}) + \Phi(-1/\bar{\tau})]\Lambda_{deg}\begin{bmatrix}\vec{f}\\ \vec{f}\end{bmatrix} \tag{3.34}$$

Using the relation

$$\Lambda_{\rm deg}\begin{bmatrix}\vec{f}\\ \vec{f}\end{bmatrix}(\tau) = \Lambda_{\rm deg}\begin{bmatrix}\vec{0}\\ \vec{f}\end{bmatrix}(ST\tau) \tag{3.35}$$

and observing that $\mathcal{F}_2^0 = TS\mathcal{F}_2^-$ one can map the problem back to the previous case

$$\begin{aligned}
\mathcal{I}_{\rm deg}\begin{bmatrix}\vec{f}\\ \vec{f}\end{bmatrix} &= \int_{\mathcal{F}_2^-} \frac{d^2\tau}{\tau_2^2}[\Phi(TS\bar{\tau}) + \Phi(STS\bar{\tau})]\Lambda_{\rm deg}\begin{bmatrix}\vec{0}\\ \vec{f}\end{bmatrix}(\tau,\bar{\tau}) \\
&= \sqrt{\det G}\int_{\rm strip} \frac{d^2\tau}{\tau_2^2}\left(\frac{\vartheta_3^4}{\vartheta_2^4} - \frac{\vartheta_4^4}{\vartheta_2^4}\right)(\bar{\tau})\sum_{\vec{m}\in\mathbb{Z}+\vec{h}} e^{-\frac{\pi}{\tau_2}\vec{m}G\vec{m}} \\
&= \frac{\sqrt{\det G}}{\pi}\sum_{\vec{m}\in\mathbb{Z}+\vec{h}} (\vec{m}G\vec{m})^{-1}
\end{aligned} \tag{3.36}$$

As we will momentarily see, this contribution has a clear counterpart at disk level in the Type I dual description.

In the third case, $\vec{h} = 0$, $\Lambda_{\rm deg}\begin{bmatrix}\vec{f}\\ \vec{0}\end{bmatrix}$ is invariant under Γ_2^+ that includes STS in the quotient Γ_2^+/Γ_2. The relevant integrals are

$$\mathcal{I}_{\rm deg}\begin{bmatrix}\vec{f}\\ \vec{0}\end{bmatrix} = \int_{\mathcal{F}_2} \frac{d^2\tau}{\tau_2^2}\Phi(\bar\tau)\Lambda_{\rm deg}\begin{bmatrix}\vec{f}\\ \vec{0}\end{bmatrix} = \int_{\mathcal{F}_2^+} \frac{d^2\tau}{\tau_2^2}[\Phi(\bar\tau)+\Phi(STS\,\bar\tau)]\Lambda_{\rm deg}\begin{bmatrix}\vec{f}\\ \vec{0}\end{bmatrix}(\tau,\bar\tau) \qquad (3.37)$$

Using the relation

$$\Lambda_{\rm deg}\begin{bmatrix}\vec{f}\\ \vec{0}\end{bmatrix}(\tau) = \Lambda_{\rm deg}\begin{bmatrix}\vec{0}\\ \vec{f}\end{bmatrix}(S\,\tau) \qquad (3.38)$$

and observing that $\mathcal{F}_2^+ = S\mathcal{F}_2^-$ one can map the problem back to the previous case

$$\begin{aligned}
\mathcal{I}_{\rm deg}\begin{bmatrix}\vec{f}\\ \vec{0}\end{bmatrix} &= \int_{\mathcal{F}_2^-} \frac{d^2\tau}{\tau_2^2}[\Phi(S\bar\tau)+\Phi(ST\bar\tau)]\Lambda_{\rm deg}\begin{bmatrix}\vec{0}\\ \vec{f}\end{bmatrix}(\tau,\bar\tau)\\
&= \sqrt{\det G}\int_{\rm strip}\frac{d^2\tau}{\tau_2^2}\left(\frac{\vartheta_2^4}{\vartheta_3^4}-\frac{\vartheta_2^4}{\vartheta_4^4}\right)(\bar\tau)\sum_{\vec{m}\in\mathbb{Z}+\vec{h}} e^{-\frac{\pi}{\tau_2}\vec{m}G\vec{m}}\\
&= 0 \qquad (3.39)
\end{aligned}$$

The vanishing result for $\vec{f}_{13} = 0, \vec{f}_{12} \neq 0$ is consistent with the fact that no perturbative type I correction to the four-hyperini Fermi interaction will be found with this tensor structure.

3.1.3 Non degenerate orbit

For simplicity we take[7] $\vec{f} = (0,0,0,f)$. The representative for these orbits may be taken to be

$$\begin{aligned}
n^i &= 0 \qquad & i &= 1,\dots p\\
m^i &< n^i \qquad & i &= p+1,\dots 4
\end{aligned} \qquad (3.40)$$

for some p. They corresponds to the cases when $\det(M_{ij}) = 0$ for $i,j \leq p$. Any other M in these orbits can be reached by a Γ_2 transformation. Conversely, one can fix the representative and act on τ with Γ_2 to enlarge the region of integration to the full upper half plane $\mathbb{C}^+$. Notice that Γ_2 transformations preserve the integer/half-integer nature of $(\vec{n},\vec{m})$.

The resulting integral is of the form

$$\mathcal{I}_{\rm ndeg}\begin{bmatrix}\vec{f}\\ \vec{h}\end{bmatrix} = \sqrt{\det G}\sum_M e^{2\pi i\mathcal{T}(M)}2^{4-1}\int_{\mathbb{C}^+}\Phi(\bar\tau)e^{-\frac{\pi\mathcal{T}_2(M)}{\tau_2\mathcal{U}_2(M)}|\tau-\mathcal{U}(M)|^2} \qquad (3.41)$$

[7] A general $\vec{f}$ can be always put into this form using an $\mathrm{SL}(4,\mathbb{Z})$ transformation on the T^4-moduli

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where the factor 2^3 accounts for the choices of signs of n^i, m^i for $i \neq \alpha$ and the sum runs over

$$
\begin{aligned}
& n^i = 0 && i = 1, \dots p \\
& n^i \in \mathbb{Z} + f^i && i = p+1, \dots 4 \\
& m^i \in \mathbb{Z} + h^i && i = 1 \dots p \\
& m^i \in \mathbb{Z} \bmod n^i + h^i && i = p+1, \dots 4
\end{aligned} \tag{3.42}
$$

Expanding $\Phi(\bar{\tau})$ in a power series

$$
\Phi(\bar{\tau}) = \sum_{\nu} c_\nu \bar{q}^\nu \tag{3.43}
$$

one can perform the gaussian integration on τ_1 that yields

$$
\begin{aligned}
\mathcal{I}_{\rm ndeg} \begin{bmatrix} \vec{f} \\ \vec{h} \end{bmatrix} = 2^3 \sqrt{\det G} \sum_M e^{2\pi i \mathcal{T}(M)} \frac{\mathcal{U}_2(M)^{\frac{1}{2}}}{\mathcal{T}_2(M)^{\frac{1}{2}}} \sum_\nu c_\nu e^{-2\pi i \nu \mathcal{U}_1(M)} \times \\
\int_0^\infty \frac{d\tau_2}{\tau_2^{3/2}} e^{-\frac{\pi \mathcal{T}_2(M)}{\tau_2 \mathcal{U}_2(M)} [\tau - \mathcal{U}_2(M)]^2} e^{2\pi\nu\tau_2 - \pi\tau_2\nu^2 \frac{\mathcal{U}_2(M)}{\mathcal{T}_2(M)}}
\end{aligned} \tag{3.44}
$$

The integral over τ_2 can be computed with the aid of

$$
\int_0^\infty \frac{dy}{y^{3/2}} e^{-ay - \frac{b}{y}} = \sqrt{\frac{\pi}{b}} e^{-2\sqrt{ab}} \tag{3.45}
$$

and yields

$$
\begin{aligned}
\mathcal{I}_{\rm ndeg} \begin{bmatrix} \vec{f} \\ \vec{h} \end{bmatrix} &= \sqrt{\det G} \sum_M \frac{2^3}{\mathcal{T}_2(M)} e^{2\pi i \mathcal{T}(M)} \sum_\nu c_\nu e^{-2\pi i \nu \bar{\mathcal{U}}(M)} \\
&= \sqrt{\det G} \sum_M \frac{2^3}{\mathcal{T}_2(M)} e^{2\pi i \mathcal{T}(M)} \Phi(\bar{\mathcal{U}}(M))
\end{aligned} \tag{3.46}
$$

Not surprisingly the result is given by a sort of generalized Hecke operator acting on the Γ_2-modular form Φ of weight zero. In the next section we will argue that the same sum appears in the ED1-instanton sum in type I theory.

$\boldsymbol{f = h}$ case. Formula (3.46) describes the generic case when hyperini come from four (in general different) fixed points. However, a significant simplification takes place for particular choices of the fixed points.

- $\vec{f} = \vec{h} = 0$. All four hyperini located at the same fixed point. With the aid of (3.29) one finds

$$
\begin{aligned}
\mathcal{I}_{\rm ndeg} \begin{bmatrix} \vec{0} \\ \vec{0} \end{bmatrix} &= \int_{\mathcal{F}_2} \frac{d^2\tau}{\tau_2^2} \Phi(\bar{\tau}) \Lambda_{\rm ndeg} \begin{bmatrix} \vec{0} \\ \vec{0} \end{bmatrix} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \sum_{s=1}^{6} \Lambda_{\rm ndeg} \begin{bmatrix} \vec{0} \\ \vec{0} \end{bmatrix} (\tau_s, \bar{\tau}_s)\, \Phi(\bar{\tau}_s) \\
&= 3 \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \Lambda_{\rm ndeg} \begin{bmatrix} \vec{0} \\ \vec{0} \end{bmatrix} = 3 \sum_M \frac{2^3 \sqrt{\det G}}{\mathcal{T}_2(M)} e^{2\pi i \mathcal{T}(M)}
\end{aligned} \tag{3.47}
$$

- $\vec{f} = \vec{h} \neq 0$. Hyperini in conjugate pairs located at two different fixed points.

  Now one finds

$$
\begin{aligned}
\mathcal{I}_{\rm ndeg}\begin{bmatrix}\vec{f}\\ \vec{f}\end{bmatrix} &= \int_{\mathcal{F}_2}\frac{d^2\tau}{\tau_2^2}\Phi(\bar\tau)\Lambda_{\rm ndeg}\begin{bmatrix}\vec{f}\\ \vec{f}\end{bmatrix} = \int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\sum_{s=1}^{6}\Lambda_{\rm ndeg}\begin{bmatrix}\vec{f}\\ \vec{f}\end{bmatrix}\Phi(\bar\tau_s)\\
&= \int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\left(\Lambda_{\rm ndeg}\begin{bmatrix}\vec{f}\\ \vec{f}\end{bmatrix} + \Lambda_{\rm ndeg}\begin{bmatrix}\vec{f}\\ \vec{0}\end{bmatrix} + \Lambda_{\rm ndeg}\begin{bmatrix}\vec{0}\\ \vec{f}\end{bmatrix}\right)\\
&= \int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\left(\Lambda_{\rm ndeg}\begin{bmatrix}\vec{0}\\ \vec{0}\end{bmatrix}(G',B') - \Lambda_{\rm ndeg}\begin{bmatrix}\vec{0}\\ \vec{0}\end{bmatrix}(G,B)\right)\\
&= 2^3\sqrt{\det G}\sum_M\left(\frac{e^{2\pi i\mathcal{T}'(M)}}{\mathcal{T}_2'(M)} - \frac{e^{2\pi i\mathcal{T}(M)}}{\mathcal{T}_2(M)}\right)
\end{aligned}
\tag{3.48}
$$

  where in the third line we have rewritten the $\mathbb{Z}_2$ shifted lattice sum in terms of the unshifted lattice sum with rescaled metric and two form

$$
G'_{ij} = 2^{-|h^i|-|h^j|}G_{ij} \qquad B'_{ij} = 2^{-|h^i|-|h^j|}B_{ij} \tag{3.49}
$$

4. Type I amplitude

Unlike in the heterotic case, the Type I dilaton belongs to a hypermultiplet and therefore the hypermultiplet geometry could receive corrections both at the perturbative and non-perturbative (ED-string) level. Yet some scattering amplitudes may vanish in perturbation theory (at least in the low-energy limit) and only receive contributions when non-perturbative effects are taken into account. For instance

$$
\mathcal{A}^{Type\ I}_{4Fermi} = \langle V^{\zeta}_{\mathbf{16}}V^{\zeta}_{\mathbf{16}^*}V^{\zeta}_{\mathbf{16}}V^{\zeta}_{\mathbf{16}^*}\rangle \tag{4.1}
$$

vanishes to all orders in perturbation theory when the external states are all located at different fixed points $f_i \neq f_j\ \forall i,j$ because of global $\mathrm{U}(1)^{16}_{\mathrm{D5}}$ symmetry. When fixed points are equal in pairs or even all equal (there are no other possibilities for $\mathbb{Z}_2$) there is a perturbative contribution that should match – and indeed does so — with the contribution of the 'degenerate orbit' in the heterotic description. In the following we will compute perturbative (disk) and ED1-instanton contributions to the four-hyperini Fermi interaction in the Type I description.

4.1 The perturbative story

In this section we compute the disk contribution in Type I theory. The disk with the four insertions of 59-string vertices has two D9 boundaries and two (in principle different) D5 boundaries. The open string amplitude reads

$$
\begin{aligned}
\mathcal{A}^{Type\ I}_{4Fermi} &= \int dx_3\langle cV^{\zeta}_{\mathbf{16}}(x_1)cV^{\zeta}_{\mathbf{16}^*}(x_2)V^{\zeta}_{\mathbf{16}}(x_3)cV^{\zeta}_{\mathbf{16}^*}(x_4)\rangle \qquad (4.2)\\
&= \zeta^{u_1}_{a_1,\bar f_1}\zeta^{\bar u_2}_{a_2,f_2}\zeta^{u_3}_{a_3,\bar f_3}\zeta^{\bar u_4}_{a_4,f_4}\,\epsilon^{a_1\ldots a_4}\left[\delta_{u_1\bar u_2}\delta_{f_2\bar f_3}\delta_{u_3\bar u_4}\delta_{f_4\bar f_1} + \delta_{u_3\bar u_2}\delta_{f_2\bar f_1}\delta_{u_1\bar u_4}\delta_{f_4\bar f_3}\right]\mathcal{A}_h
\end{aligned}
$$

The open string vertex operators are given by

$$
\begin{aligned}
V^{\zeta}_{\mathbf{16}} &= \zeta^{u}_{a,\bar f}(p)\, S^a e^{-\varphi/2} \sigma_f e^{ipX} \Lambda_{\bar f,u} \\
V^{\zeta}_{\mathbf{16}^*} &= \zeta^{\bar u}_{a,f}(p)\, S^a e^{-\varphi/2} \sigma_f e^{ipX} \bar\Lambda_{\bar u,f}
\end{aligned} \tag{4.3}
$$

Rather than the heterotic fermions λ, the Type I vertex operators involve Chan-Paton matrices $\Lambda_{u,f}$ in the bifundamental of $\mathrm{U}(16)_{\mathrm{D9}} \times \mathrm{U}(1)^{16}_{\mathrm{D5}}$ yielding

$$
\Lambda_{\bar f_1 u_1} \Lambda_{\bar u_2 f_2} \Lambda_{\bar f_3 u_3} \Lambda_{\bar u_4 f_4} = \delta_{u_1 \bar u_2} \delta_{f_2 \bar f_3} \delta_{u_3 \bar u_4} \delta_{f_4 \bar f_1} \tag{4.4}
$$

or

$$
\Lambda_{\bar f_1 u_1} \Lambda_{\bar u_4 f_4} \Lambda_{\bar f_3 u_3} \Lambda_{\bar u_2 f_2} = \delta_{u_3 \bar u_2} \delta_{f_2 \bar f_1} \delta_{u_1 \bar u_4} \delta_{f_4 \bar f_3} \tag{4.5}
$$

for the two possible cyclically inequivalent orderings. They correspond to the two tensor structures in (4.2).

The integration runs over the real line with $x_1 < x_2 < x_3 < x_4$. The correlators coincides with their counterpart in the heterotic string but now insertions lie on the boundary of the disk

$$
\begin{aligned}
\frac{\prod_i dx_i}{dV_{\mathrm{SL}(2,\mathbb{R})}} &= dx_3 \, \langle c(x_1) c(x_2) c(x_4) \rangle = dx_3 \, x_{12} x_{14} x_{24} \\
\langle \prod_{i=1}^{4} e^{ik_i X}(x_i) \rangle &= \prod_{i<j} x_{ij}^{2\alpha' k_i k_j} = x^{\alpha' s} (1-x)^{\alpha' t} \\
\langle \prod_{i=1}^{4} S^{a_i} e^{-\varphi/2}(x_i) \rangle &= \epsilon^{a_1 \ldots a_4} \prod_{i<j} x_{ij}^{-\frac{1}{2}} \\
\langle \prod_{i=1}^{4} \sigma_{f_i}(x_i) \rangle &= \mathcal{C}_{qu}(x)\, \Lambda_{cl} \begin{bmatrix} \vec 0 \\ \vec h \end{bmatrix} (x)
\end{aligned} \tag{4.6}
$$

The amplitude is expressed in terms of the $\mathrm{SL}(2,\mathbb{R})$ invariant ratio

$$
x = \frac{x_{12} x_{34}}{x_{13} x_{24}} = \frac{\vartheta_3^4(it)}{\vartheta_4^4(it)} \tag{4.7}
$$

with t the modular parameter of the annulus doubly covering the disk. The classical and quantum part of the four-twist correlator read

$$
\begin{aligned}
\mathcal{C}_{qu}(x) &= 2^{-\frac{4}{3}} \prod_{i<j} x_{ij}^{-\frac{1}{6}} \, t^{-2} \, \eta(it)^{-4} \\
\Lambda_{cl} \begin{bmatrix} \vec 0 \\ \vec h \end{bmatrix} (x) &= \sqrt{\det G} \sum_{\vec m \in \mathbb{Z} + h} e^{-\frac{\pi}{t} \vec m \cdot G \cdot \vec m}
\end{aligned} \tag{4.8}
$$

The classical part counts windings of open strings connecting D5-branes living at the fixed points 1 and 2. Plugging (4.6) and (4.8) into (4.2) one finds

$$
\mathcal{A}_h(s,t) = \int \frac{dz}{t^2 \eta(it)^4} x^{\alpha' s - \frac{2}{3}} (1-x)^{\alpha' t - \frac{2}{3}} \, \Lambda_{cl} \begin{bmatrix} \vec 0 \\ \vec h \end{bmatrix} \tag{4.9}
$$

Switching to the annulus measure via

$$\frac{dx}{x} = i\pi\vartheta_2^4(it)dt \tag{4.10}$$

and sending $s,t \to 0$ one finds

$$\begin{aligned}\mathcal{A}_h(0,0) &= \int_0^\infty \frac{dt}{t^2}\,\Lambda_{cl}\begin{bmatrix}\vec{0}\\ \vec{h}\end{bmatrix}(it) = \sqrt{\det G}\int_0^\infty \frac{dt}{t^2}\sum_{\vec{m}\in\mathbb{Z}+\vec{h}} e^{-\frac{\pi}{t}\vec{m}G\vec{m}}\\ &= \frac{\sqrt{\det G}}{\pi}\sum_{\vec{m}\in\mathbb{Z}+\vec{h}}\frac{1}{\vec{m}G\vec{m}}\end{aligned} \tag{4.11}$$

Formula (4.11) perfectly matches the dual heterotic result (3.36) coming from worldsheet instantons in degenerate orbits with tensor structure $\delta_{u_1\bar{u}_2}\delta_{u_3\bar{u}_4}$ and fixed point locations $\vec{f}_{12} = \vec{f}_{13}$! It is quite remarkable that for this particular configuration of fixed points, no other perturbative contributions beyond the disk are necessary to reproduce the degenerate orbits contributing to the heterotic amplitude. This should be a consequence of the 1/2 BPS nature of the coupling. Let us notice, however, that the heterotic contribution (3.33), though independent of B, has no disk counterpart in the type I description. Indeed, there is no room for connected disk amplitudes with the tensor structure $\delta_{u_1\bar{u}_2}\delta_{f_1f_2}\delta_{u_3\bar{u}_4}\delta_{f_3f_4}$.

4.2 ED1-brane instantons

Finally we consider non-pertubative corrections to the four-hyperini coupling in the Type I description. The dynamics of ED1-instantons is described by the two-dimensional gauge theory governing the interactions of open strings ending on ED1-branes. Here we follow [34] where the gauge theory describing D1 bound states in toroidal compactifications of Type I theory was studied in great details and its dynamics was shown to match that of the fundamental Heterotic string (see also [31–33, 62]). More precisely, the ED1 gauge theory was shown to flow in the infrared to a symmetric product CFT that can be analyzed with standard orbifold techniques. This simpler IR description was exploited to study the spectrum of BPS states of the theory. We start by reviewing the match for F^4-terms in type I/heterotic theory on T^2 [29] (here extended to T^4). Then we consider the closely relate BPS saturated four-hyperini terms in $T^4/\mathbb{Z}_2$.

4.2.1 T^4-case

The low energy dynamics of a bound state of k D1-strings in ten-dimensional Type I theory is described by an $O(k)$ two-dimensional $\mathcal{N} = (8,0)$ supersymmetric gauge theory with the following matter content [31, 32]

$$\begin{array}{rl} X^I, S^a & \mathbf{\tfrac{1}{2}k(k+1)} \\ S^{\dot{a}} & \mathbf{\tfrac{1}{2}k(k-1)} \\ \lambda^u & \mathbf{k} \end{array} \tag{4.12}$$

with $I = 1, \ldots \mathbf{8_v}$, $a = 1, \ldots \mathbf{8_s}$, $\dot{a} = 1, \ldots \mathbf{8_c}$ and $u = 1, \ldots 32$. X^I, S^a fields come from D1-D1 open strings while λ^u describe excitations of D1-D9 strings. The moduli space of the theory, defined by the flatness condition $[X^I, X^J] = 0$, is parameterized by diagonal matrices X^I with components X^I_t, $t = 1, \ldots k$ (the D1 positions). In the infrared limit off-diagonal components of the adjoint matter X^I, S^a and $S^{\dot{a}}$ can pair up and become massive. This is not the case for the D1-D9 fermions λ^u_t that remain massless since D1's move always inside D9-branes. After integrating out the massive modes, one is left with the following field content

$$X^I_t, S^a_t, \lambda^u_t \qquad t = 1, \ldots k \tag{4.13}$$

The Cartan fields in (4.13) are defined up to transformations of the Weyl group $S_k \ltimes \mathbb{Z}_2^k$ of $O(k)$ with S_k acting by permutations and $\mathbb{Z}_2$ acting by reflection on the $O(k)$-fundamentals $\lambda^u_t \to -\lambda^u_t$. The four possible $\mathbb{Z}_2$-holonomies of λ^u_t along the two cycles four spin structures of the SO(32) fermions in the heterotic string. The resulting field content is that of k copies of the Green-Schwarz heterotic string moving in the target space

$$\mathcal{M}: \qquad (\mathbb{R}^6 \times T^4)^k / S_k \tag{4.14}$$

with light-cone coordinates chosen along a two-cycle inside T^4. The Type I / Heterotic dictionary can be further tested by considering F^4-couplings in the two descriptions. Insertions of F^4 can soak at most 8 fermionic zero modes and therefore this coupling will receive contributions only from 1/2 BPS excitations of the orbifold CFT. Notice that only even spin structures of λ^u_t can contribute to this amplitude since λ^u_t in the odd spin structure contribute additional fermionic zero modes. This is in contrast with the recently found instanton generated superpotentials in orientifold braneworlds [1 − 22] coming from λ-like fermionic zero modes appearing in the two-dimensional ED1-gauge theory only in the odd-spin structure.

The string vertex operator for F can be derived from the D1-D9 action and is given by

$$V_F = F^a_{\mu\nu} \frac{U_2}{k} \int d^2 z \sum_{t=1}^{k} (X^\mu_t \partial X^\nu_t - S_t \gamma^{\mu\nu} S_t)(z) \lambda_t T^a \lambda_t \, (\bar{z}) \tag{4.15}$$

Notice that the right and left moving part of the vertex operator are given by the currents for SO(8) transverse Lorentz group and SO(32) gauge group. Choosing F along the Cartan of these groups the string amplitude can be written in the simple form

$$\langle V_F^4 \rangle = \frac{U_2^4}{k^4} \partial_v^4 \partial_{\bar{v}}^4 \, \langle e^{v \cdot J_{\mathrm{SO}(8)} + \bar{v} \cdot J_{\mathrm{SO}(32)}} \rangle \big|_{v=\bar{v}=0} = \frac{U_2^4}{k^4} \partial_v^4 \partial_{\bar{v}}^4 \, \mathcal{X}^{\mathrm{D1}}(v, \bar{v}) \big|_{v=\bar{v}=0} \tag{4.16}$$

with $\mathcal{X}(v, \bar{v})$ the weigthed partition function and $v, \bar{v}$ belonging to the Cartan of SO(8) and SO(32) respectively.

In the following we evaluate $\mathcal{X}^{\mathrm{D1}}(v, \bar{v})$ in the symmetric product CFT. We restrict ourselves to 1/2-BPS contributions $\mathcal{X}^{\mathrm{D1}}_{\mathrm{BPS}}(v, \bar{v}) \sim v^4$ since contributions from states preserving less supersymmetries (higher orders in v) will cancel from (4.16).

1/2-BPS states come from sectors in the symmetric product CFT with exactly 8 fermionic zero modes. We recall that twisted sectors of the symmetric product CFT are classified by the conjugacy classes of S_k

$$[g]: \qquad (1)^{m_1}(2)^{m_2}\ldots\ldots(k)^{m_k} \tag{4.17}$$

with $\sum_\ell \ell\, m_\ell = k$ and (ℓ) referring to a cyclic permutation of length ℓ. Each factor $(\ell)^{m_\ell}$ can be thought as m_ℓ copies of "short" strings of length ℓ. States in this sector are projected by the centralizer

$$\mathcal{C} = \prod_\ell S_{m_\ell} \ltimes \mathbb{Z}_\ell^{m_\ell} \tag{4.18}$$

It is easy to check that the center of mass of any short string group $(\ell)^{m_\ell}$ in (4.17) is invariant under the orbifold projection and therefore leads to 8 fermionic zero modes. The only twisted sectors with exactly 8 fermionic zero modes are then those with a single $[g] = (\ell)^m$ factor with $m\ell = k$. This has to be projected by $(m)\times\mathbb{Z}^s_{\ell,\mathrm{diag}}$ with $s = 0,\ldots \ell-1$. The weigthed BPS partition $\chi^{\mathrm{D1}}_{\mathrm{BPS}}(v,\bar v)$ can then be evaluated by tracing over states in the twisted sectors labelled by ℓ, m, s.

We consider a D1-string wrapping on a T^2-cycle of T^4 specified by the two vectors $M_k = (\vec k_1, \vec k_2)$ each made out of four integers (with greatest common divisor 1). $\vec k_{1,2}$ count the number of times the two 1-cycles of T^2 wind around the four 1-cycles of T^4. The induced Kähler $\mathcal{T}(M_k)$ and complex structures $\mathcal{U}(M_k)$ of this T^2-cycle are given by (3.23). Now take $k = m\ell$ ED1-strings wrapping this two-cycle. Labelling fields in the symmetric product CFT by Φ_{i,t_i} with $i = 1,\ldots m$, $t_i = 1,\ldots \ell$. The generic $\left[(\ell)^m, (m)\times\mathbb{Z}^s_{\ell,\mathrm{diag}}\right]$-twisted boundary conditions can be written as

$$\begin{aligned}\Phi_{t_i,i}(\sigma_1+1,\sigma_2) &= \Phi_{t_i+1,i}(\sigma_1,\sigma_2)\\ \Phi_{t_i,i}(\sigma_1+U_1,\sigma_2+U_2) &= \Phi_{t_i+s,i+1}(\sigma_1,\sigma_2)\end{aligned} \tag{4.19}$$

Gluing the k-copies, one can form a single periodic field on a torus with Kähler structure $k\mathcal{T}(M_k)$ and complex structure $[m\mathcal{U}(M_k)+s]/\ell$. Using (3.23) the results can be written in the suggestive form

$$\begin{aligned}\mathcal{U}(\ell,m,s,M_k) &= \frac{m\,\mathcal{U}(M_k)+s}{\ell} = \mathcal{U}(M)\\ \mathcal{T}(\ell,m,s,M_k) &= \ell m\,\mathcal{T}(M_k) = \mathcal{T}(M)\end{aligned} \tag{4.20}$$

with

$$M = M_k\cdot\begin{pmatrix}\ell & s\\ 0 & m\end{pmatrix} \tag{4.21}$$

Summarizing the 1/2-BPS index $\mathcal{X}^{\mathrm{D1}}_{\mathrm{BPS}}$ for k D1-strings can be written in terms of the $k=1$ D1 (i.e. the heterotic) partition function $\mathcal{X}_{\mathrm{het}}$ with modular parameter $\mathcal{U}(M)$, weighted by the ED1-action $2\pi\mathcal{T}(M)$. More precisely

$$\mathcal{X}^{\mathrm{BPS}}_{D1}(v,\bar v,T,U) = \sum_{k,M_k}\sum_{m\ell=k}\sum_{s=0}^{\ell-1} e^{2\pi i k T(M_k)}\mathcal{X}_{\mathrm{het}}\left(mv, m\bar v; \frac{m\bar{\mathcal{U}}(M_k)+s}{\ell}\right) \tag{4.22}$$

with

$$\mathcal{X}_{\rm het}\,(v,\bar v;\tau) = \frac{\sqrt{\det G}}{\tau_2^4}\,\frac{\theta_1(v)^4}{\eta^{12}} \sum_{\alpha=2,3,4} \frac{\bar\theta_\alpha^{16}(\bar v)}{\bar\eta^{24}} \tag{4.23}$$

Plugging (4.22) into (4.16) one finds

$$\begin{aligned} \langle V_F^4 \rangle &= \sum_{k,M_k} \sum_{m\ell=k} \sum_{s=0}^{\ell-1} e^{2\pi i k \mathcal{T}(M_k)}\, \mathcal{X}^{\rm BPS}_{\rm het} \left(\frac{m\bar{\mathcal{U}}(M_k)+s}{\ell} \right) \\ &= \sum_M e^{2\pi i \mathcal{T}(M)}\, \mathcal{X}^{\rm BPS}_{\rm het}\left(\bar{\mathcal{U}}(M)\right) \end{aligned} \tag{4.24}$$

with

$$\mathcal{X}^{\rm BPS}_{\rm het}\,(\bar\tau) = \partial_v^4 \partial_{\bar v}^4\, \mathcal{X}_{\rm het}\,(v,\bar v,\bar\tau)\,\big|_{v=\bar v=0} = \frac{\sqrt{\det G}}{\bar\eta^{24}} \begin{cases} \sum_\alpha \bar\theta_\alpha^{15} \bar\theta_\alpha'''' \\ \sum_\alpha \bar\theta_\alpha^{14} (\bar\theta_\alpha'')^2 \end{cases} \tag{4.25}$$

The two cases correspond to taking either all four F equal or pairwise different elements of the SO(32) Cartan subgroup. The result (4.24) precisely matches the contribution of non-degenerate orbits in the heterotic string computed in [29] !

4.2.2 $T^4/\mathbb{Z}_2$-case

One can extend the type I / Heterotic dictionary described in the last section to $T^4/\mathbb{Z}_2$. The instanton dynamics is now governed by a gauge theory describing the excitations of unoriented strings connecting ED1, D5, and D9 branes. We focus again on the infrared dynamics where only the fields along the Cartan are relevant.

Let us describe the fermionic content of this theory (bosonic fields follow from the residual $\mathcal{N} = (4,0)$ worldsheet supersymmetry). Take a single $k = 1$ ED-string wrapped on a two-cycle inside T^4. The $\mathbb{Z}_2$ projects out 4 of the 8 ED1-ED1 fermionic modes S^a. The surviving $a = 1,\ldots,4$ runs over the spinor representation of SO(5,1). In addition the ED1-ED9 modes λ's transform with eigenvalues "i". This can be seen by noticing that $\mathbb{Z}_2$ reflects the gauge vector (since the ED1 wraps $T^4/\mathbb{Z}_2$) that couples in the ED1-D9 action linearly to $\lambda\lambda$. This is precisely the $\mathbb{Z}_2$-action in the λ heterotic fermions and breaks SO(32) down to U(16). Finally ED1-D5 open strings lead to one extra fermionic mode μ_{f_i} at each of the 16 fixed points. Indeed ED1-D5 open strings have 8 ND directions and therefore the only massless excitation of this string comes from the fermionic Ramond ground state.

A disk computation involving three twisted and one untwisted vertices shows that the hyperini ζ^u_{a,f_i} couple to all these fermionic modes via the four-fermi coupling

$$\mathcal{L}_{4f} = \zeta^u_{a,f} \mu_f \lambda_u S^a + \zeta^{\bar u}_{a,f} \bar\mu_f \lambda_{\bar u} S^a \tag{4.26}$$

The four-hyperini coupling in the effective action can be computed by bringing down four powers of $\mathcal{L}_{4f}$ in the gauge theory path integral. Notice that this precisely soaks up the four fermionic zero modes S^a leading to a non-trivial result. The resulting correlator is given by the insertion of $V_\zeta = \mu_f \lambda_u S^a$ in the gauge path integral. This vertex has the right $\mathrm{SO}(5,1)\times\mathrm{U}(16)_{\rm D9}$ quantum numbers to be identified with the heterotic twisted vertex (3.2) at fixed point f. This suggests that the infrared dynamics of the $k = 1$ ED1-D5-D9 system can be described by the heterotic sigma model on $\mathbb{R}^6 \times T^4/\mathbb{Z}_2$, trivially generalizing the T^4 result of the previous section. Notice that unlike in the T^4 case the ED1 world-sheet wraps a curved two-cycle $T^2/\mathbb{Z}_2$ inside K3. This makes the ED1/heterotic dictionary less transparent and requires a Green-Schwarz formulation of the heterotic theory with light-cone coordinates chosen along the curved space-time two-cycle wrapped by the instanton. Still, resorting to the covariant description, (3.2) is the operator in the CFT with the correct quantum numbers supporting our proposal. This suggests that the infrared dynamics of the $k=1$ ED1-D5-D9 system can be described by the heterotic sigma model on $\mathbb{R}^6 \times T^4/\mathbb{Z}_2$, trivially generalizing the T^4 result of the previous section[8].

For $k>1$ one finds k copies of the heterotic string moving on the symmetric product target space $(\mathbb{R}^6 \times T^4/\mathbb{Z}_2)^k/S_k$. We can now proceed like in the T^4-case. The two-cycle wrapped by the $k=1$ D-string inside T^4 is specified by $M_k = (\vec{k}_1, \vec{k_2})$ with $\vec{k}_{1,2}$ made out of integers with greatest common divisor 1. The induced Kahler and complex structure of this two-cycle is $\mathcal{T}(M_k)$ and $\mathcal{U}(M_k)$ respectively. Using the fact that only twisted sectors with exactly four fermionic zero modes of S^a contribute to the amplitude we restricted to $(\ell)^m$-twisted sectors projected by $\mathbb{Z}^s_{\ell,\mathrm{diag}}$. This folds the k copies into a single field on a worldsheet with Kähler and complex structure $\mathcal{T}(M) = k\,\mathcal{T}(M_k)$ and $\mathcal{U}(M) = [m\,\mathcal{U}(M_k)+s]/\ell$. This is in perfect agreement with the heterotic result (3.46) for the four-hyperino coupling on $T^4/\mathbb{Z}_2$. Indeed using (4.21) one can always rewrite the matrix M describing the wrapping number of the heterotic string worldsheet in terms of a matrix M_k specifying the two-cycle inside T^4 that the ED1 wraps and the integers ℓ, m, s, with $m\ell = k$ and $s = 0, \ldots \ell-1$, specifying the twisted sector in the k-symmetric product orbifold CFT.

5. Outlook

In the present paper we exploited Heterotic / Type I duality on $T^4/\mathbb{Z}_2$ to improve our understanding of stringy multi-instanton calculus in theories with 8 supercharges. In particular we computed the four-hyperini Fermi interaction

$$S_{4Fermi} = \int d^6x d^4\theta W_{f_1 f_2 f_3 f_4} \prod_{i=1}^{4} H_{16,f_i} \tag{5.1}$$

with $H_{16,f}$ the twisted hypermultiplet matter transforming in the $\mathbf{16}_{-3}, \mathbf{16}^*_{+3}$ of U(16). This amplitude is tree-level exact at the perturbative level but it receives corrections from the infinite sum of 1/2 BPS worldsheet instantons connecting the fixed points f_i.

In the type I side, perturbative corrections are allowed only for particular choices of the four fixed points where the hyperini are located. In the cases where such perturbative corrections are present at disk level we matched the result against the contribution of degenerate orbits in the heterotic description. When the four hyperini are located at different fixed points the coupling is absent at any order in perturbation theory but generated non-perturbatively via ED1-instantons wrapping two cycles on $T^4/\mathbb{Z}_2$ connecting the four

[8] More precisely, the transverse degrees of freedom live on $\mathbb{R}^6 \times (T^2/\mathbb{Z}_2)_\perp$ while the light-cone coordinates are compactified on $(T^2/\mathbb{Z}_2)_\parallel$.

fixed points. This is the $D = 6$ analog of the stringy generated superpotentials in $\mathcal{N} = 1$ theories recently studied in [1−22].

The rules for multi D-brane instanton counting in this setting properly generalize those for toroidal compactifications. It is beyond the scope of the present investigation to tackle the subtle problem of the field theory interpretation of these stringy or exotic instantons. A list of possible candidates are Hyper-instantons [63], Octonionic instantons [64], Euclidean monopoles [65] or IC-instantons [66].

Finally, it is worth commenting on a somewhat unexplored mechanism for moduli stabilization for Type I and other unoriented theories like the one presently considered. It consists in the Higgsing of anomalous U(1)'s living on D-branes sitting at orbifold fixed points thanks to their mixing with (twisted) RR axions. The efficiency of this mechanism in the present model is perfectly clear in the Heterotic description where all such states (vectors, axions and their superpartners) are absent altogether from the massless spectrum[9]. In $D = 4$ a remnant of the $D = 6$ anomaly are massive (non-)anomalous U(1)'s [67, 68] that may have interesting generalized Chern-Simons couplings [69−71]. It would be worth exploring further the effectiveness of (non)anomalous U(1) in removing axions from the massless spectrum in a supersymmetric fashion and thus 'stabilizing' the corresponding moduli superfields. We could then start talking about a *Petite Bouffe*!

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A. Modular group and its subgroups

In this appendix we collect some relevant information on the modular group $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ and its finite index subgroups of order 2.

The modular group. The modular group $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ is a infinite discrete group. $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ acts on the upper half plane $\mathbb{C}^+$ ($\tau_2 > 0$) by projective transformations

$$\tau \to \frac{a\tau + b}{c\tau + d} \quad \text{with} \quad a, b, c, d \in Z : ad - bc = 1 \tag{A.1}$$

Under projective transformations

$$\tau_2 \to \frac{\tau_2}{|c\tau + d|^2} \tag{A.2}$$

[9] M. B. would like to thank E. Kiritsis, F. Quevedo, B. Schellekens and A. Uranga for interesting discussions and comments on this point.

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Its fundamental region $\mathcal{F} = \mathbb{C}^+/\Gamma$ is

$$\mathcal{F} = \{\tau : \tau_2 > 0, |\tau_1| < 1/2, |\tau|^2 > 1\} \tag{A.3}$$

$\Gamma = \mathrm{SL}(2,\mathbb{Z})$ is generated by the two transformations T and S

$$T \;:\; \tau \to \tau + 1 \quad , \quad S \;:\; \tau \to -1/\tau \tag{A.4}$$

T and S satisfy

$$S^2 = (-)1 \quad , \quad (ST)^3 = 1 \tag{A.5}$$

$\mathcal{F} = \mathbb{C}^+/\Gamma$ contains one cusp point $\tau = i\infty$ (a 'fixed point' of the parabolic element T) and two elliptic points $\tau = i$ (of order 2, fixed under S) and $\tau = \exp(2\pi i/3) \approx \exp(\pi i/3)$ (of order 3, fixed under ST).

$\mathrm{SL}(2,\mathbb{Z})$ is the discrete version of the global conformal group on the sphere $\mathrm{SL}(2,C)$ or better its restriction to the disk $\mathrm{SL}(2,R)$. With this in mind it is important to observe that a modular transformation with $c \neq 0$ can be conveniently written in the form

$$(c\tau' - a) = -\frac{1}{c\tau + d} \tag{A.6}$$

Transformations with $c = 0$ are simply T^ℓ i.e. rigid translations. The above rewriting makes it clear that modular transformations map circles centered at $-d/c$ on the τ_1 axis into circles centered at a/c on the τ_1' axis (which actually coincides with the τ_1 axis as a whole). For $c = 0$ circles become vertical lines. In particular T maps circle into circle and vertical lines into vertical lines. S maps circles of radius R centered in the origin into circles of radius $1/R$ centered in the origin. Moreover vertical lines $\tau_1 = r$ are mapped by S into circles of radius $1/2r$, centered at $\tau_1 = -1/2r$ and thus passing through the origin, and *vice versa*. These observations help following the maps of the (equivalent) copies of the fundamental region $\mathcal{F}$ (known as 'hyperbolic triangles', each containing a cusp and two elliptic points).

For our purposes it is necessary to introduce some finite index subgroups of Γ: Γ_2, Γ_2^+, Γ_2^-, Γ_2^0.

Γ_2 modular subgroup. Γ_2 is the group of projective transformations with

$$\Gamma_2 \;:\; b, c = 0 \bmod 2 \qquad \text{thus } a, d = 1 \bmod 2 \tag{A.7}$$

Γ_2 is generated by T^2 and ST^2S. Γ_2 is the subgroup of Γ preserving all half-integer spin structures. Its fundamental region has genus 0 and is obtained from $\mathcal{F} = \mathbb{C}^+/\Gamma$ by the action of the elements of the coset Γ/Γ_2 i.e.

$$\mathcal{F}_2 = \mathbb{C}^+/\Gamma_2 = \{1, T, S, TS, ST, TST\}\mathcal{F} \tag{A.8}$$

A convenient representation for $\mathcal{F}_2$ is the region bounded by the vertical lines $\tau_1 = \pm 1$ and by the two circles $(\tau_1 \mp 1/2)^2 + \tau_2^2 = 1/4$.

This in particular means that Γ_2 is of index 6 in Γ and contains 3 cusps at $\tau = 0, 1, i\infty$ and no fixed points.

Γ_2^- modular subgroup. Γ_2^- is the group of projective transformations with

$$\Gamma_2 \ : \quad c = 0 \bmod 2 \quad \text{ thus } a, d = 1 \bmod 2 \tag{A.9}$$

Γ_2^- is generated by T and ST^2S. Γ_2^- is the subgroup of Γ preserving the half-integer spin structure of ϑ_2 i.e. $[^0_h]$. Its fundamental region has genus 0 and is obtained from $\mathcal{F} = \mathbb{C}^+/\Gamma$ by the action of the elements of the coset Γ/Γ_2^- i.e.

$$\mathcal{F}_2^- = \mathbb{C}^+/\Gamma_2^- = \{1, S, STS\}\mathcal{F} \tag{A.10}$$

A convenient representation for $\mathcal{F}_2^-$ is the region bounded by the vertical lines $\tau_1 = \pm 1/2$ and by the two circles $(\tau_1 \mp 1/2)^2 + \tau_2^2 = 1/4$. This in particular means that Γ_2^- is of index 3 in Γ and contains 2 cusps at $\tau = 0, i\infty$ and one fixed points of order 2 at $\tau = (1+i)/2$.

Γ_2^+ modular subgroup. Γ_2^+ is the group of projective transformations with

$$\Gamma_2 \ : \quad b = 0 \bmod 2 \quad \text{ thus } \quad a, d = 1 \bmod 2 \tag{A.11}$$

Γ_2^+ is generated by STS and ST^2S. Γ_2^+ is the subgroup of Γ preserving the half-integer spin structure of ϑ_4 i.e. $[^f_0]$. Its fundamental region has genus 0 and is obtained from $\mathcal{F} = \mathbb{C}^+/\Gamma$ by the action of the elements of the coset Γ/Γ_2^+ i.e.

$$\mathcal{F}_2^+ = \mathbb{C}^+/\Gamma_2^+ = \{1, S, T\}\mathcal{F} \tag{A.12}$$

A convenient representation for $\mathcal{F}_2^+$ is the region bounded by the vertical lines $\tau_1 = \pm 1$ and by the two circles $(\tau_1 \mp 1)^2 + \tau_2^2 = 1$.

This in particular means that Γ_2^+ is of index 3 in Γ and contains 2 cusps at $\tau = 0, i\infty$ and one fixed points of order 2 at $\tau = 1 + i$.

Γ_2^0 modular subgroup. Γ_2^0 is the group of projective transformations with

$$\Gamma_2^0 \ : \ b + c = 0 \bmod 2 \quad \text{and } a + b = 1 \bmod 2 \tag{A.13}$$

Γ_2^0 is generated by S and ST^2S. Γ_2^0 is the subgroup of Γ preserving the half-integer spin structure of ϑ_3 i.e. $[^f_f]$.

A convenient representation for $\mathcal{F}_2^0$ is the region bounded by the vertical lines $\tau_1 = \pm 1$ and by the unit circle $(\tau_1)^2 + \tau_2^2 = 1$.

Its fundamental region has genus 0 and is obtained from $\mathcal{F} = \mathbb{C}^+/\Gamma$ by the action of the elements of the coset Γ/Γ_2^0 i.e.

$$\mathcal{F}_2^0 = \mathbb{C}^+/\Gamma_2^0 = \{1, T, TS\}\mathcal{F} \tag{A.14}$$

This in particular means that Γ_2^0 is of index 3 in Γ and contains 2 cusps at $\tau = \pm 1, i\infty$ and one fixed points of order 2 at ($\tau = i$ under S).

It proves crucial for our manipulations to observe that the above fundamental regions are related to one another:

$$\mathcal{F}_2^- = S\mathcal{F}_2^+ \quad , \quad \mathcal{F}_2^+ = T\mathcal{F}_2^0 \quad , \quad \mathcal{F}_2^+ = TS\mathcal{F}_2^- \tag{A.15}$$

where $T^2 = 1$ (true for weak modular forms of Γ_2^α with $\alpha = \pm, 0$ or nothing) as well as $S^2 = 1$ has been used.

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